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Linear and Quadratic Spline Interpolation at Knot Averages

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For linear and quadratic spline interpolation at nodes which are weighted avearges of the spline knots, conditions on the weights are developed which guarantee that the corresponding spline interpolation operators are norm-bounded independently of the knot locations.

1. INTRODUCTION

Let $k \geqslant 0$ and $n \geqslant 1$ be integers and let

$$\mathbf{t}\colon t_1 < t_2 < \cdots < t_{n+k+1}$$

be a strictly increasing sequence of finite real numbers. A function s with domain $[t_1, t_{n+k+1}]$ is said to be a spline of degree k with knots $\mathbf{t}$ provided

(a) $s \in C^{k-1}[t_1, t_{n+k+1}]$ (if $k \geqslant 1$),

(b) $s^{(j)}(t_1) = s^{(j)}(t_{n+k+1}) = 0$ for $j = 0, 1, \ldots, k-1$ (if $k \geqslant 1$) and

(c) $s(x)$ is a polynomial of degree k or less on each interval (t_i, t_{i+1}) for $i = 1, 2, \ldots, n+k$.

If $k = 0$, $s(t_i) = \frac{1}{2}[s(t_i+) + s(t_i-)]$ for $i = 1, 2, \ldots, n+1$. For fixed $\mathbf{t}$ and k the set of such splines is a linear vector space whose dimension is n. The end conditions (b) have been chosen for convenience below. In practical applications they are often modified to suit the conditions of a particular problem. In most such instances the assertions below will still be valid.

Let $\mathbf{w}\colon w_0, w_1, \ldots, w_k, w_{k+1}$ be a sequence of nonnegative numbers whose sum is one with

$$w_0 < 1 \qquad \text{and} \qquad w_{k+1} < 1. \tag{1.1}$$

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Let $\boldsymbol{\tau}: \tau_1 < \tau_2 < \cdots < \tau_n$ be defined by

$$\tau_i = w_0 t_i + w_1 t_{i+1} + \cdots + w_{k+1} t_{i+k+1}.$$

Let $\mathbf{y}: y_1, y_2, ..., y_n$ be an arbitrary sequence of real numbers. A spline function s of degree k with knots $\mathbf{t}$ is said to interpolate the data $\mathbf{y}$ at the nodes $\boldsymbol{\tau}$ if

(d) $s(\tau_i) = y_i$ for $i = 1, 2, ..., n$.

For fixed k, $\mathbf{w}$, and $\mathbf{t}$ the association from data vectors $\mathbf{y}$ to their interpolating splines $s = P\mathbf{y}$ defines the splines interpolation operator P. The constraint (1.1) on w_0 and w_{k+1} is sufficient (and necessary) to guarantee that P is well-defined.

The operator norm $\|P\|$ is defined as

$$\|P\| = \sup\{\|P\mathbf{y}\|: \|\mathbf{y}\| = 1\},$$

where

$$\|\mathbf{y}\| = \max_i |y_i| \qquad \text{and} \qquad \|s\| = \max_x |s(x)|.$$

The purpose of this article is, for some low values of k, to derive some conditions on $\mathbf{w}$ which will guarantee that $\|P\|$ is bounded independently of $\mathbf{t}$.

The following has already appeared in [6].

THEOREM 1.1. *For fixed degree $k \geqslant 1$ and fixed $\mathbf{w}$ a necessary condition that $\|P\|$ be bounded independently of $\mathbf{t}$ is that*

$$w_1 > 0, w_2 > 0, ..., w_k > 0.$$

Proof. Let $n = 4$, let $\mathbf{y}$ be given by $y_i = (-1)^i$, and let C be a bound on $\|P\|$. Since

$$\tau_3 - \tau_2 = w_0(t_3 - t_2) + w_1(t_4 - t_3) + \cdots + w_{k+1}(t_{4+k} - t_{k+3})$$

we may "highlight" a specific w_j by letting

$$t_2, t_3, ..., t_{2+j} \to -1 \qquad \text{and} \qquad t_{3+j}, t_{4+j}, ..., t_{4+k} \to +1.$$

This coalescence is known (see [3]) to preserve the first-derivative continuity of $s = P\mathbf{y}$ if $0 < j < k + 1$. But then the mean-value theorem and a theorem of A. A. Markov (see [7]) gives, in the limit,

$$2 = s(\tau_2) - s(\tau_3) = s'(\varepsilon)(\tau_2 - \tau_3) \leqslant Ck^2(\tau_3 - \tau_2) = 2Ck^2 w_j$$

so that $w_j \geqslant 1/(Ck^2) > 0$. ∎

A corollary of this theorem is the well-known fact that cubic spline interpolation "at the knots" ($\tau_i = t_{i+2}$) is not uniformly norm-bounded.

2. The Cases $k = 0$ and $k = 1$

For $k = 0$, $\|P\| = 1$ independently of $\mathbf{t}$ (and of $\mathbf{w}$).

For $k = 1$ the problem is no longer trivial. We have

THEOREM 2.1. *For spline interpolation of degree one* $\limsup_{\mathbf{t}} \|P\|$ *is finite if and only if* $w_0 < \frac{1}{2}$ *and* $w_2 < \frac{1}{2}$ *in which case*

$$\limsup_{\mathbf{t}} \|P\| = \max\{1/(1-2w_0),\, 1/(1-2w_2)\}.$$

Proof. We may assume that each open knot interval (t_j, t_{j+1}) for $j = 2, 3,..., n$ includes at least one of the τ_i since otherwise it would be possible, for each $\mathbf{y}$, to delete either the knots $t_1, t_2,..., t_{j-1}$ or the knots $t_{j+2},..., t_{n+1}, t_{n+2}$ (and some of the data) without changing the value of $\|P\mathbf{y}\|$. With this assumption, all but one of the nodes have been located. The remaining node appears on some closed interval $[t_p, t_{p+1}]$ with $1 \leqslant p \leqslant n+1$.

The following assumptions involve some loss of generality: We assume that $p < n+1$ and that $\|P\mathbf{y}\| = |s(t_{p+r})|$ with $r > 0$. Setting $\tau_0 = t_1$ if necessary, we have

$$t_p \leqslant \tau_{p-1} < \tau_p \leqslant t_{p+1} < \tau_{p+1} < t_{p+2} < \cdots < \tau_{p+r-1} < t_{p+r}.$$

Without any further loss of generality, we maximize $\|P\mathbf{y}\|$ by assuming that $p > 1$ and that $s(\tau_i) = (-1)^i$ for each i. For convenience we set $a_i = |s(t_{p+i})|$. Similar triangles give

$$\frac{a_1 + 1}{2} = \frac{t_{p+1} - \tau_{p-1}}{\tau_p - \tau_{p-1}}$$

$$= \frac{(w_0 + w_1)(t_{p+1} - t_p) + w_0(t_p - t_{p-1})}{w_2(t_{p+2} - t_{p+1}) + w_1(t_{p+1} - t_p) + w_0(t_p - t_{p-1})} < \frac{w_0 + w_1}{w_1}$$

and

$$\frac{a_i + a_{i-1}}{1 + a_{i-1}} = \frac{t_{p+i} - t_{p+i-1}}{\tau_{p+i-1} - t_{p+i-1}},$$

$$= \frac{t_{p+i} - t_{p+i-1}}{(w_1 + w_2)(t_{p+i} - t_{p+i-1}) + w_2(t_{p+i+1} - t_{p+i})}$$

$$< \frac{1}{w_1 + w_2} \qquad \text{for} \quad i = 2, 3,..., r.$$

These inequalities solve recursively as

$$a_j < \frac{1}{1-2w_0}\left[1 - \frac{2w_0(w_0-w_2)}{w_1}\left(\frac{w_0}{1-w_0}\right)^{j-1}\right]$$

if $w_0 \neq \frac{1}{2}$. They solve as

$$a_j < \frac{1+2jw_1-w_1}{w_1}$$

if $w_0 = \frac{1}{2}$.

An argument on the relatives sizes of adjoining knot intervals shows that all of these inequalities are sharp so that we must conclude that $\|P\mathbf{y}\| = a_r$ is bounded independently of r if and only if $w_0 < \frac{1}{2}$, in which case the best bound is $\|P\mathbf{y}\| < 1/(1-2w_0)$.

The loss of generality mentioned above resulted from our assumptions that $p < n+1$ and $r > 0$. If these are relaxed, we have an exactly symmetric instance which leads to the best bound $\|P\mathbf{y}\| < 1/(1-2w_2)$. ∎

COROLLARY 2.2. *For spline interpolation of degree one,* $\|P\| = 1$ *independently of* $\mathbf{t}$ *if and only if* $w_0 = w_2 = 0$.

If, as is sometimes fashionable, one adds the local-mesh-ratio constraint

$$0 < q \leqslant \frac{t_{i+1}-t_i}{t_i-t_{i-1}} \leqslant 1/q \qquad \text{for} \quad i = 2, 3, \ldots, n+1$$

the effect is to replace w_0 by $w_0 - qw_2$ and to replace w_2 by $w_2 - qw_0$ in the statements of Theorem 2.1 provided that q is sufficiently small that these quantities are positive.

3. THE CASE $k = 2$

For quadratic spline interpolation we will produce a family of necessary conditions on $\mathbf{w}$ that $\|P\|$ be bounded independently of $\mathbf{t}$ by considering the particular choice of knots

$$t_j = 1 + q + q^2 + \cdots + q^{j-2} \qquad \text{for} \quad j = 1, 2, \ldots, n+3 \tag{3.1}$$

and requiring that $\|P\|$ be bounded independently of n for each $q > 0$. This contrasts to the approach in [6], where q was permitted to go to zero "first."

Our theorem will involve the following three sets of conditions on $\mathbf{w}$, each parametrized by θ.

CONDITION A. For $1 - w_0 < \theta \leqslant 1$

$$2(q_1 + 1)\,\theta^2 > 1 - q_1 + 4q_1\theta, \tag{3.2}$$

where

$$q_1 = \frac{2(\theta - 1 + w_0)}{(w_2 + w_3) + \sqrt{(w_2 + w_3)^2 + 4w_3(\theta - 1 + w_0)}}. \tag{3.3}$$

Condition B. For $1 < \theta < \theta_1$

$$2(q_1 + 1)(\theta - 1)^2 < q_1^2 - q_1^3 + 4q_1^2(\theta - 1), \tag{3.4}$$

where q_1 is given by (3.3) and

$$\theta_1 = \frac{1 - w_2 + w_3 + \sqrt{(w_0 + w_1)^2 + 4w_0 w_3}}{2w_3} \tag{3.5}$$

if $w_3 > 0$, while $\theta_1 = \infty$ if $w_3 = 0$.

Condition C. For $1 - w_3 < \theta \leqslant 1$

$$2(q_2 + 1)\,\theta^2 > 1 - q_2 + 4q_2\theta, \tag{3.6}$$

where

$$q_2 = \frac{2(\theta - 1 + w_3)}{(w_0 + w_1) + \sqrt{(w_0 + w_1)^2 + 4w_0(\theta - 1 + w_3)}}. \tag{3.7}$$

Observe that Condition C is the symmetric form of Condition A.

THEOREM 3.1. *For quadratic spline interpolation with* **t** *giuven by* (3.1), *a necessary and sufficient condition that* $\|P\|$ *be bounded independently of* n *for each* $q > 0$ *is that* **w** *satisfy* $w_1 > 0$, $w_2 > 0$, *and Conditions* A, B, *and* C.

Proof. We begin by deducing Condition A. If $w_0 = 0$, (3.2) is a triviality. Thus, we suppose $w_0 > 0$. We set

$$\theta = \theta(q) = 1 - w_0 + (w_2 + w_3)\,q + w_3 q^2. \tag{3.8}$$

Then, in view of (3.1),

$$\begin{aligned}\tau_i - t_i &= (1 - w_0)(t_{i+1} - t_i) + (w_2 + w_3)(t_{i+2} - t_{i+1}) + w_3(t_{i+3} - t_{i+2}) \\ &= \theta(t_{i+1} - t_i) \qquad \text{for} \quad i = 1, 2, ..., n.\end{aligned}$$

If $1 - w_0 < \theta < 1$, as it is for sufficiently small q, then a quadratic spline $s(x)$ can be represented for $t_2 \leqslant x \leqslant t_4$ as

$$s(x) = s(t_2)\left[\frac{(x-t_3)(x-\tau_2)}{(t_3-t_2)(\tau_2-t_2)} - \frac{(\tau_3-\tau_2)(x-t_3)_+^2}{(t_3-t_2)(\tau_2-t_2)(\tau_3-t_3)}\right]$$
$$+ s(t_3)\left[\frac{(x-t_2)(x-\tau_2)}{(t_3-t_2)(t_3-\tau_2)} - \frac{(\tau_3-t_2)(\tau_3-\tau_2)(x-t_3)_+^2}{(t_3-t_2)(t_3-\tau_2)(\tau_3-t_3)^2}\right]$$
$$+ s(\tau_2)\left[\frac{(\tau_3-t_2)(x-t_3)_+^2}{(\tau_2-t_2)(t_3-\tau_2)(\tau_3-t_3)} - \frac{(x-t_2)(x-t_3)}{(\tau_2-t_2)(t_3-\tau_2)}\right]$$
$$+ s(\tau_3)\frac{(x-t_3)_+^2}{(\tau_3-t_3)^2},$$

where $(x - t_3)_+ = \max\{x - t_3, 0\}$. By setting $x = t_4$ and simplifying, we have

$$s(t_4) + as(t_3) + bs(t_2) = [s(\tau_3) + qs(\tau_2)]/\theta^2$$

with

$$a = \frac{1 + 2q\theta - (1+q)\,\theta^2}{\theta^2} \qquad \text{and} \qquad b = \frac{q(1-\theta)^2}{\theta^2}. \tag{3.9}$$

Indeed the recurrence

$$s(t_{i+2}) + as(t_{i+1}) + bs(t_i) = [s(\tau_{i+1}) + qs(\tau_i)]/\theta^2 \tag{3.10}$$

holds for $i = 1, 2,..., n-1$. From the geometry of parabolas it is clear that we can maximize $|s(t_n)|$ subject to the constraints $|s(\tau_i)| = |y_i| \leqslant 1$ by setting

$$s(\tau_i) = (-1)^i \qquad \text{for} \quad i = 1, 2,..., n. \tag{3.11}$$

Solving (3.10) for this data and requiring that $s(t_n)$ be bounded independently of n gives

$$|s(t_i)| = C_1(1 - \beta_1^{i-1}) + C_2(1 - \beta_2^{i-1}) \qquad \text{for} \quad i = 1, 2,..., n+1, \tag{3.12}$$

where $0 < \beta_2 < \beta_1 < 1$ are solutions of $\beta^2 - a\beta + b = 0$. (It is necessary, but elementary, to ascertain that $a^2 - 4b > 0$.) The constants C_1 and C_2 are determined by the initial conditions

$$s(t_2) = \frac{-1}{\theta^2} \qquad \text{and} \qquad s(t_3) = \frac{a + 1 - q}{\theta^2}. \tag{3.13}$$

The condition that $\beta_1 < 1$ is equivalent to $1 - a + b > 0$ or

$$2(q+1)\,\theta^2 > 1 - q + 4q\theta. \tag{3.14}$$

Combining (3.8) and (3.14) yields the inequality (3.2) of Condition A for $1 - w_0 < \theta < 1$. Letting θ tend to 1 then yields $q_1 \leqslant 1$.

A separate argument, which we omit, shows that this inequality must be strict. This finishes the proof of Condition A is necessary.

By symmetry Condition C is also necessary.

To prove that Condition B is necessary we choose q so that $t_{i+1} < \tau_i < t_{i+2}$ and, hence, $1 < \theta < 1 + q$. Since this causes us to replace $\tau_i - t_i = \theta(t_{i+1} - t_i)$ by $\tau_{i-1} - t_i = (\theta - 1)(t_{i+1} - t_i)/q$ in our representations of $s(x)$, the effet is to replace θ by $(\theta - 1)/q$ and to replace $s(\tau_i)$ by $s(\tau_{i-1})$ throughout the above discussion. Another change is that the two initial conditions (3.13) are replaced by "mixed boundary conditions" $s(t_3) + (a - q)\, s(t_2) = q^2 s(\tau_1)/(\theta - 1)^2$ and $(aq - b)\, s(t_{n+2}) + bqs(t_{n+1}) = q^4 s(\tau_n)/(\theta - 1)^2$. The replacement for (3.12) is

$$|s(t_i)| = C_3 + C_4 \beta_0^i + C_5 \beta_1^i \qquad \text{for} \quad i = 2, 3, \ldots, n + 2, \tag{3.15}$$

where $0 < \beta_1 < \beta_0$ are solutions of $\beta^2 - a\beta + b = 0$ with a and b now given by

$$a = \frac{q^2 + 2q^2(\theta - 1) - (1 + q)(\theta - 1)^2}{(\theta - 1)^2} \quad \text{and} \quad b = \frac{q(q + 1 - \theta)^2}{(\theta - 1)^2}. \tag{3.16}$$

Because C_4 includes the factor β_0^{-n}, the boundedness requirement is now $\beta_1 < 1 < \beta_0$ which is equivalent to the inequality (3.4) of Condition B. Since the restriction $\theta < \theta_1$ is equivalent to $\theta < 1 + q_1$ with θ_1 and q_1 given by (3.5) and (3.3), respectively, the proof of necessity for Condition B is now complete.

To prove sufficiency for these three conditions we first note that the steps of the above argument are reversible. We also need to observe that: (1) the data given by $y_i = (-1)^i$ are essentially "worst-case" and (2) $\|s\|$ is bounded independently of n for each $\theta \neq 1, \theta_1$ whenever the sets $\{s(t_i)\}$ and $\{s(\tau_i)\}$ are bounded independently of n. These observations being granted and a separate argument being supplied for the cases $\theta = 1, \theta_1$, the remaining details in the proof of sufficiency are straightforward. ∎

It is easy to exploit Conditions A, B, C for any specific **w**. In addition some special choices for θ give useful information about general choices of **w**. For example, letting θ tend to $1 - w_0$ gives $2(1 - w_0)^2 \leqslant 1$. This is implied by results in [6]. Another example concerns $\theta = 1$ which yields $w_0 < w_2 + 2w_3$. An example which gives no information is the **w**, θ combination for which (accidentally?) $q_1 = 2\theta - 2$; Condition B becomes the tautology $1 < 2$.

For other perspectives on quadratic spline interpolation, see [2, 4, 5, 8, 9].

4. Remark

The author is indebted to the referee for pointing out the following corollary of Theorem 2.1:

THEOREM. *If K is any nonempty compact subset of the set of points (w_0, w_1, w_2) satisfying $w_0 + w_1 + w_2 = 1$, $w_i \geqslant 0$, $w_0 < \frac{1}{2}$, $w_2 < \frac{1}{2}$, and if the points of interpolation $\tau_i = w_0 t_i + w_1 t_{i+1} + w_2 t_{i+2}$ with $\mathbf{w} \in K$, but possibly depending on t_i, t_{i+1}, t_{i+2}, are used for spline interpolation of degree $k = 1$, then* $\lim \sup_{\mathbf{t}} \|P\|$ *is finite.*

It would have been nice if the conditions of Theorem 3.1 were easy enough to use to verify the quadratic analogue of the above.

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